Mathematica Moravica
Vol. **6** (2002), 87–92

FIXED POINT THEOREMS FOR MAPPINGS IN d-COMPLETE TOPOLOGICAL SPACES

Valeriu Popa

Abstract. Fixed point theorems are given for pairs of mappings satisfying an implicit relation defined on d-complete topological spaces.

1. Introduction

Let (X, τ) be a topological space and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y.X$ is said to be d-complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence x_n is convergent in (X, τ). Complete metric spaces and quasicomplete metric spaces are examples of d-complete topological spaces. If d satisfies and $d(x, y) = d(y, x)$, $\forall x, y \in X$, then d is a symmetric on X. Recently, Hicks [1], Hicks and Rhoades [2] and Saliga [5] proved several fixed point theorems in d-complete topological spaces. Let $T : X \to X$ be a mapping, T is ω-continuous at x if $x_n \to x$ implies $Tx_n \to Tx$ as $n \to \infty$.

The following family of real functions was introduced by M.A.Khan, M.S.Khan and S,Sessa in [4]. Let ϕ denote the family of all real functions $\phi : R_+^3 \to R_+$ satisfying the following conditions:
(C_1): ϕ is lower semi-continuous in each coordinate variable,
(C_2). Let $v, w \in R_+$ such that $v \geq \phi(w, v, w)$ or $v \geq \phi(w, w, v)$.
Then $v \geq h.w$, where $h = \phi(1, 1, 1) > 1$.

In [5] Saliga proved the following.

Theorem 1. *Let (X, τ, d) be a d-complete topological spaces where d is a continuous symmetric. Let A, B map C, a closed subset of X, into (onto) X such that $C \subset A(C)$, $C \subset B(C)$ and*

$$d(Ax, By) \geq g\big(d(x, y), d(Ax, x), d(By, y)\big) \quad \textit{for all } x,\ y \textit{ in } C$$

where $g \in \phi$. Then A and B have a common fixed point in C.

The purpose of this paper is to prove some fixed point theorems which generalize Theorem 1 and others for mappings satisfying an implicit relation.

AMS (MOS) Subject Classification 1991. Primary: 54H25.
Key words and phrases: d-topological space, fixed point.

2. Implicit relations.

Let $F(t_1,\dots,t_4): R_+^4 \to R$ be a continuous mapping.We define the following properties:
(H_1): There exists $h>1$ such that for every $u\geq 0$, $v\geq 0$ with $F(u,v,u,v)\geq 0$ or $F(u,v,v,u)\geq 0$ we have $u\geq hv$.
(H_2): There exists $h>1$ such that for every $u>0$, $v>0$ with $F(u,v,u,v)\geq 0$ or $F(u,v,v,u)\geq 0$ we have $u\geq hv$.
(H_3): $F(0,v,0,v)\geq 0$ or $F(0,v,v,0)\geq 0$ implies $v=0$.
(H_u): $F(u,u,0,0)<0, \forall u>0$.

Ex.1.

$$F(t_1,\dots t_4)=t_1-(at_2^2+bt_3^2+ct_4^2+dt_1t_3)^{1/2}$$

where $a>1$, $0<c<1$ and $0<b+d<1$.
(H_1): Let $u\geq 0$, $v\geq 0$ and $F(u,v,u,v)=u-(av^2+bu^2+cv^2+du^2)\geq 0$. Then $u\geq (\dfrac{a+c}{1-b-d}^{1/2}\cdot v=h_1v$, where $h_1=\dfrac{a+c}{1-b-d}^{1/2}>1$. Let $u\geq 0$, $v>0$ and $F(u,v,v,u)=u-(av^2+bv^2+cu^2+duv)^{1/2}\geq 0$ which implies $u^2(1-c)-duv-v^2(a+b)\geq 0$. Then $f(t)=t^2(1-c)-dt-(a+b)\geq 0$ where $t=\dfrac{u}{v}$. Since $f(0)<0$ and $f(1)<0$ then there exists $h_2>1$ such that $f(h_2)=0$ and $f(t)\geq 0$ for $t\geq h_2$, thus $u\geq h_2v$. If $v=0$ then $u\geq h_2v$. For $h=\min\{h_1,h_2\}$ it follows that $u\geq hv$.
(H_u): $F(u,u,0,0)=u(1-a^{1/2}<0$, $\forall u>0$.

Ex.2.

$$F(t_1,\dots,t_4)=t_1\max\{t_1,t_3,t_4\}-t_2(at_2+bt_3+ct_4)$$

where $a>1$, $b,c\geq 0$.
(H_2) Let $u>0$, $v>0$ be and let $F(u,v,u,v)=u\max\{u,v\}-v(av+bu+cv)\geq 0$.

If $v\geq u$ then $uv(1-a-b-c)\geq 0$, a contradiction. Then $u>v$ and $u^2-v^2(a+b+c)\geq 0$ which implies $u\geq hv$, where $h=(a+b+c)^{1/2}>1$.

Similary, if $F(u,v,v,u)\geq 0$ then $u\geq hv$.
(H_3): $F(0,v,0,v)=-(a+c)v\geq 0$ implies $v=0$. Similary, $F(0,v,v,0)\geq 0$ implies $v=0$
(H_u): $F(u,u,0,0)=u^2(1-a)<0$, $\forall u>0$.

Ex.3.

$$F(t_1,\dots,t_4)=t_1\max\{t_1,t_2,t_4\}-at_2\min\{t_2,t_3,t_4\}$$

where $a>1$.
(H_2): Let $u>0$, $v>0$ and $F(u,v,u,v)=\max\{u,v\}-av\min\{u,v\}\geq 0$. If

$v \geq u$ then $uv(1-a) \geq 0$, a contradiction. Thus $u > v$ and $u \geq hv$, where $h = a^{1/2} > 1$.
(H_3): $F : (0, v, 0, v) = 0$ and H_3 is not satisfied.
(H_4): $F(u, u, 0, 0) = u^2 > 0$, $\forall u > 0$ and (H_u) is not satisfied.

3. Main results

Theorem 2. *Let C be a subset of a d-topological space (X, τ, d) and suppose that A, B map C into(onto) X such that $C \subset A(C)$, $C \subset B(C)$. Moreover,we assume that*

$$F(d(Ax, By), d(x, y), d(x, Ax), d(y, By)) \geq 0 \tag{1.1}$$

for all x, y in X, where F satisfies condition (H_u). Then A and B have at most one common fixed point.

Proof. Suppose that A and B have two common fixed points z, z' with $z \neq z'$. Then by (1) we have succesively

$$F(d(Az, Bz'), d(z, z')d(z, Az), d(z', Bz')) \geq 0$$
$$F(d(z, z'), d(z, z'), 0, 0) \geq 0,$$

a contradiction of (H_u).

Theorem 3. *Let (X, τ, d) be a d-complete topological space where d is a continuous symmetric. Let A and B map C,a closed subset of X into(onto) X such that $C \subset A(C)$, $C \subset B(C)$. If A, B satisfies the inequality (1) for all x, y in C where F satisfies condition (H_1) Then A and B have a common fixed point. Further, if F satisfies in addition condition (H_u) then the common fixed point is unique.*

Proof. Let $x_0 \in C$ Since $C \subset A(C)$ there exists $x_1 \in C$ such that $Ax_1 = x_0$. Now $C \subset B(C)$ so there exists $x_2 \in C$ such that $Bx_2 = x_1$. Build the sequence x_n by $Ax_{2n+1} = x_{2n}$, $Bx_{2n+2} = x_{2n+1}$. Now, if $x_{2n+1} = x_{2n}$ for some n, the x_{2n+1} is a fixed point of A. Then, by (1) we have successively

$$F(d(Ax_{2n+1}, Bx_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, Ax_{2n+1}), d(x_{2n+2}, Bx_{2n+2})) \geq 0$$
$$F(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1})) \geq 0$$
$$F(0, d(x_{2n+1}, x_{2n+2}), 0, d(x_{2n+2}, x_{2n+1})) \geq 0.$$

By (H_1) $0 \geq h.d(x_{2n+1}, x_{2n+2})$. Therefore $x_{2n+1} = x_{2n+2}$ and $Bx_{2n+1} = Bx_{2n+2} = x_{2n+1}$. Therefore x_{2n+1} is a common fixed point of A and B. Now if $x_{2n+1} = x_{2n+2}$ for some n, then by (1) we have succesively

$$F(d(Ax_{2n+3}, Bx_{2n+2}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+3}, Ax_{2n+3}), d(x_{2n+2}, Bx_{2n+2})) \geq 0$$

$$F(d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+3}, x_{2n+2}), d(x_{2n+2}, x_{2n+1})) \geq 0$$

$$F(0, d(x_{2n+2}, x_{2n+3}), d(x_{2n+3}, x_{2n+2}), 0) \geq 0.$$

By (H_1) $0 \geq h.d(x_{2n+3}, x_{2n+2})$. Hence $x_{2n+2} = x_{2n+3}$. Thus $Ax_{2n+2} = Ax_{2n+3} = x_{2n+2}$ and x_{2n+2} is a fixed point of A also.

Suppose that $x_n \neq x_{n+1}$ for all n. By (1) we have succesively

$$F(d(Ax_{2n+1}, Bx_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, Ax_{2n+1}), d(x_{2n+2}, Bx_{2n+2})) \geq 0$$

$$F(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2})) \geq 0$$

Again by (H_1) we have

$$d(x_{2n+1}, x_{2n}) \geq h.d(x_{2n+1}, x_{2n+2})$$

or

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{h}.d(x_{2n}, x_{2n+1}).$$

Also

$$F(d(Ax_{2n+3}, Bx_{2n+2}), d(x_{2n+3}, x_{2n+2}), d(x_{2n+3}, Ax_{2n+3}), d(x_{2n+2}, Bx_{2n+2})) \geq 0$$

By (H_1) we get

$$d(x_{2n+2}, x_{2n+3}) \leq \frac{1}{h}.d(x_{2n+1}, x_{2n+2}).$$

By induction gives

$$d(x_{n+1}, x_{n+2}) \leq (\frac{1}{h})^n.d(x_0, x_1).$$

Thus

$$\sum_{n=1}^{\infty} d(x_{n+1}, x_{n+2}) \leq \sum_{n=1}^{\infty} (\frac{1}{h})^{n+1} d(x_0, x_1)$$

X is d-complete so $x_n \to p$ where $p \subset C$, since C is closed. We also have $x_{2n} \to p$ and $n \to \infty$. This gives $Ax_{2n+1} \to p$ and $Bx_{2n+2} \to p$ as $n \to \infty$. Since $p \in C$, $p \in A(C)$ and $p \in B(C)$, so there exist $v, w \in C$ such that $Av = p$ and $Bw = p$. Now

$$F\big(d(Ax_{2n+1}, Bw), d(x_{2n+1}, w), d(x_{2n+1}, Ax_{2n+1}), d(Bw, w)) \geq 0$$

Since F is continuous, letting $n \to \infty$ gives

$$F\big(0, d(p,w), 0, d(p,w)\big) \geq 0 \tag{1.2}$$

and by (H_1) we have $0 \geq h.d(p,w)$. Hence $p = w$. Also

$$F(d(Av, Bx_{2n+2}), d(v, x_{2n+2}), d(Av, v), d(x_{2n+2}, Bx_{2n+2})) \geq 0$$

Letting $n \to \infty$ gives

$$F(0, d(v,p), d(v,p), 0) \geq 0 \tag{1.3}$$

and by (H_1) $0 \geq h.d(p,v)$. Hence $p = v$. Therefore, $Ap = Av = p = Bv = Bp$.

If F satisfies condition (H_u) by Theorem 2 p is the unique common fixed point of A and B.

Theorem 4. *Let (X, τ, d) be a d-complete topological space where d is a continuous symmetric. Let A and B map C, a closed subset of X, into(onto) X such that $C \subset A(C)$, $C \subset B(C)$. If A and B satisfies inequality (1) for all x, y in C, where F satisfies condition (H_2) and (H_3), then A and B have a common fixed point.*

Further, if F satisfies in addition condition (H_u), then the common fixed point is unique.

Proof. As in Theorem 3 x_n is a Cauchy sequence and so $\lim x_n = p$. Since $p \in C$, $p \in A(C)$ AND $p \in B(C)$, so that exist $v, w \in C$ such that $Av = p$ and $Bw = p$. As in Theorem 3 we have (2) which implies by (H_3) that $d(p,w) = 0$. As in Theorem 3 we have (3) which implies by (H_3) that $v = p$. Therefore $Ap = Av = p = Bv = Bp$. If F satististies condition (H_u) by Theorem 2 p is the unique common fixed point of A and B.

Theorem 5. *Let (X, τ, d) be a d-complete topological Hausdorff space where d is a continuous symmetric. Let A and B map C, a closed subset of X, into(onto) X, such that $C \in A(C)$, $C \in B(C)$. If A and B satisfies inequality (1) for all x, y in C, where F satisfies condition (H_2) and A and B are continuous, then A and B have a common fixed point.*

Proof. As in Theore 3 x_n is a Cauchy sequence and so has an unique limit p. By the continuity of

$$A, A(p) = A(\lim x_{2n+1}) = \lim Ax_{2n+1} = \lim x_n = p.$$

Similary $B(p) = p$.

4. References

[1] Hicks, T. L.: *Fixed point theorems for d-complete topological spaces*, I. Internat. J. Math. and Math. Sci., **15**(1992), 435-440.

[2] Hicks, T. L. and Rhoades B. E.: *Fixed point theorems for d-complete topological spaces II*, Math.Japonica, **37**(1992), 847-853.

[3] Hicks, T. L. and Saliga: *Fixed point theorems for non-self maps II*, Math.Japonica, **30**(1993), 953-956.

[4] Khan, M. A., Khan, M. S. and Sessa, S.: *Some theorems on expansion mappings and their fixed points*, Demonstratio Math. **19**(1986), 673-683.

[5] Saliga, L. M.: *Fixed point theorems for non-self maps in d-complete topological spaces*, Internat.J.Math.Sci. **19**(1996), 103-110.

Department of Mathematics
University of Bacău
5500 Bacău
Romania
E-mail:vpopa@ub.ro